Proof of the Equivalent Area of a Circle and a Right Triangle with Leg Lengths of the Radius and Circumference

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Abstract

In this paper I seek to prove Archimedes' Theorem that a circle of radius r and circumference C has the same area as a right triangle with legs of lengths r and C. I will use a cut up and rearrange method to show that the circle may be rearranged into a parallelogram which may then be rearranged into the triangle.

1 Introduction

In Archimedes' *Measurement of the Circle*, he proposes that a circle is equal in area to a right triangle, K, with legs of length of the radius and circumference of the circle. His original proof uses a method of contradiction to show that the area of polygons inscribed within the circle which have areas less than the triangle are always smaller than the circle and polygons circumscribing the circle will have areas greater than the triangle and circle. Showing that the area of the circle is equal to the triangle (Archimedes).

In this paper, I will seek to prove Archimedes' Proposition 1 through rearrangement and limit rules. In Section 2 I will prove the lemmas and theorems backing my rearrangement lemma. In Section 3 I will demonstrate graphically my method of rearrangement.

2 Necessary Proofs

In order to prove Archimedes proposition, I will first need to prove several necessary lemmas which I will use. First, I will show that a right triangle may be cut up to assemble a rectangle. I will then show that any parallelogram with base b and height h has the same area. Next, I will show that as the number

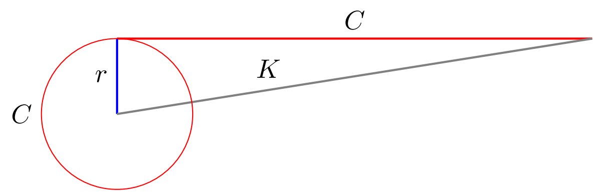

Figure 1: Archimedes' Theorem

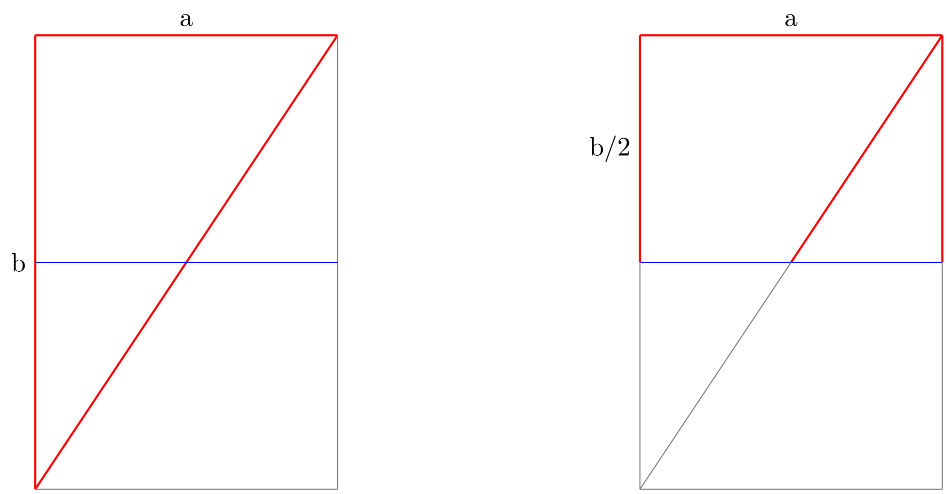

Figure 2: Triangle with Equal Area to Rectangle

of sectors cut from a disk increase toward infinity, a triangle may accurately be used to approximate each sector.

Lemma 1. *A right triangle with legs of length a and b may be cut to form a rectangle of equal area with sides of length a and $\frac{b}{2}$.*

Proof. Let us begin by defining a right triangle with legs of length a and b. Now let us create another right triangle with those same leg lengths and place it's hypotenuse on that of the first triangle. Now we have created a rectangle with sides of length a and b, with twice the area of the original triangle as there are two triangles composing the rectangle. Now, to half the area to return to the area of the original triangle, let us draw a midpoint on the legs of both triangles' legs of length b. Now draw a line between those points and divide the rectangle in half as shown in figure 2. Each half of the rectangle is equal in area to the original triangle with sides of length a and b.

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Lemma 2. *Any parallelogram with a height of h and a base of length b has the*

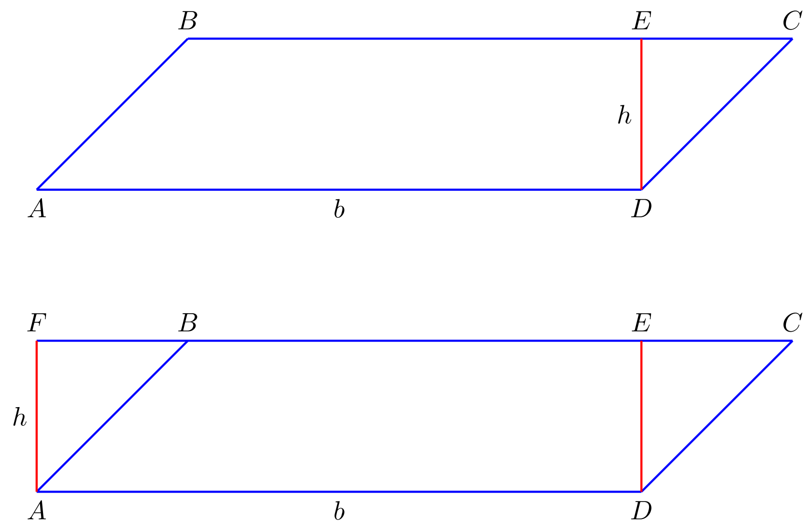

Figure 3: Parallelogram with Equal Area to Rectangle

same area.

Proof. Any parallelogram with a height h and base b has an equal are to a rectangle of sides h and b. We will begin using the parallelogram $ABCD$ in figure 3. First, we will draw a line perpendicular to $\overline{AB}$ from D to a point E on $\overline{BC}$ which is the height and the length is h. The triangle $\triangle ECD$ may be moved such that the side $\overline{CD}$ would match to the side $\overline{AB}$. This will create a rectangle $FADE$ as $\overline{ED}$ was normal to the line $\overline{AD}$ so $\overline{FA}$ will be perpendicular to $\overline{FE}$ and $\overline{AD}$. Finally, since any parallelogram can be converted into a rectangle of equal area of side lengths h and b, then any parallelogram with height h and base length b must have the same area.

□

3 Graphical Proof

The goal for this graphical proof is to show that the area of the triangle is equal to the area of the circle as shown in figure 1.

To begin, I will explain a method of cutting up the circle into sectors. Next, I will place those sectors into an approximate parallelogram. Finally, I will increase the number of sectors toward infinity to create a very close approximation to a rectangle.

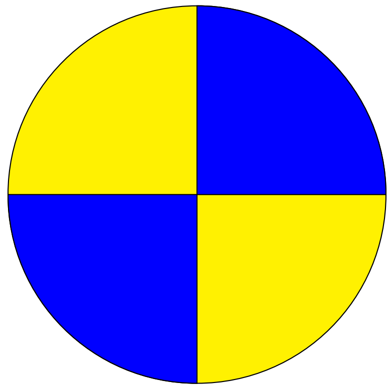

Figure 4: Sectored Circle with $n = 4$.

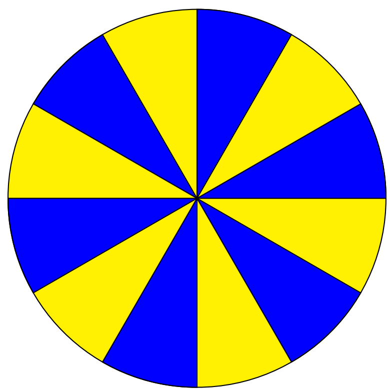

Figure 5: Sectored Circle with $n = 12$.

3.1 Dividing the Circle

I will begin my proof by dividing the circle into n sectors where n is an even natural number greater than or equal to 4. This is done by drawing n radii from the center of the circle with angles between each radii of $\frac{2\pi}{n}$ between them as in figure 4 with $n = 4$ and figure 5 with $n = 12$. I choose to define n to be even to generate parallelograms at each step toward infinity instead of of creating trapezoids for odd values of n. I state that $n \geq 4$ to prevent undefined triangles that would be formed at $n = 2$ with an straight angle.

3.2 Rearrangement

These sectors may then be bounded by a parallelogram as shown in figure 6 and 7 with $n = 4$ and $n = 12$. I say that they may be bounded as there is a

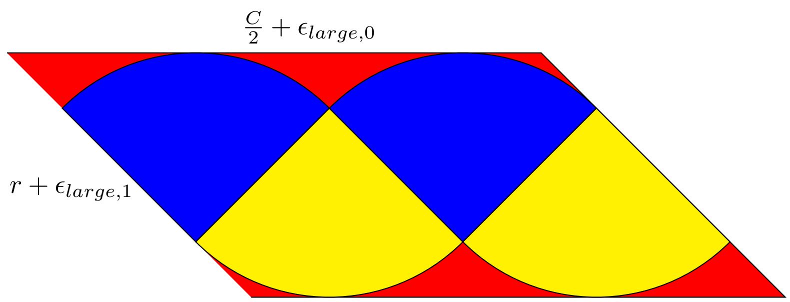

Figure 6: Rearranged Sectored Circle with $n = 4$.

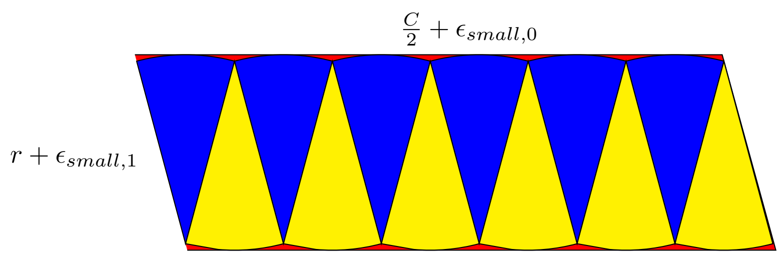

Figure 7: Rearranged Sectored Circle with $n = 12$.

greatest and smallest y value of their coordinates that may be used to define a horizontal boundary above and below the rearrangement. The left and right sides may be drawn with the angle of the sector to enclose the figure and will form parallel sides as the angle off of the vertical is equal on the leftmost sector and the rightmost. The sloped side of this figure approaches r as the number of sectors approaches infinity and the base approaches one half the circumference. We will maintain that n will be divisible by 2 to keep the parallelogram shape on to infinity. As shown in the figures, the area which is red or the area inside the enclosing parallelogram but outside the sectors from the original circle decreases as the number of sectors goes to infinity. I use $r + \epsilon_{large}$ and $r + \epsilon_{small}$ to denote that the excess length becomes smaller as more sectors are added. These hold real values which may be found through trigonometry but decrease to 0 as the number of sectors increase to infinity.

In figure 8, we let n approach infinity. At this point, the height of the parallelogram approaches r as the length of the curvature of the arc past the legs approaches 0. The base of the figure approaches the circumference halved because the arcs of the sectors become nearly linear as the number of sectors grows to infinity and the sum of arcs for half the sectors will be half the circumference. Now, we have a parallelogram with a height of h and base of $\frac{C}{2}$. By lemma 2, this means that the area is known to be the base $\frac{C}{2}$ times the height r. Using lemma 1, we can show that a right triangle with legs of length r and C may

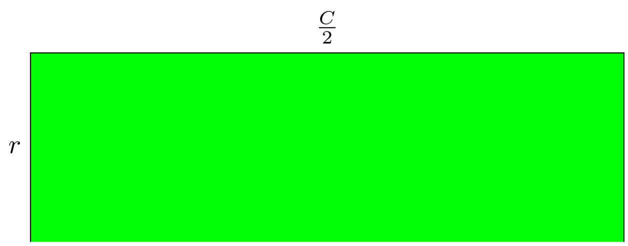

Figure 8: Rearranged Sectored Circle with $n \to \infty$.

be rearranged to form the rectangle. Finally, by Euclid's first Common Notion, since the area of the circle is equal to that of the parallelogram and the area of the triangle is equal to that of the parallelogram, both the triangle and the circle have equal areas (Euclid).

4 Bibliography

Archimedes, and Thomas Little Heath. *The Works of Archimedes.* University Press, 1897.

Euclid, and David E Joyce. "Common Notions." *Euclid's Elements*, Clark University, 1996, mathcs.clarku.edu/ djoyce/elements/bookI/cn.html.